15. Double Integrals Over a General Region Part 2

In this section, we will talk about:

- Properties of Double Integrals
- Double Integrals in Polar Coordinates

Properties of Double Integrals

Assume that all of the following integrals exist. Then,

- $\iint_D [f(x,y)+g(x,y)]dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA$
- $\iint_D cf(x,y)dA = c\iint_D f(x,y)dA$
- If $f(x,y) \geqslant g(x,y)$ for all (x,y) in D, then

$$\iint_D f(x,y)dA \geqslant \iint_D g(x,y)dA$$

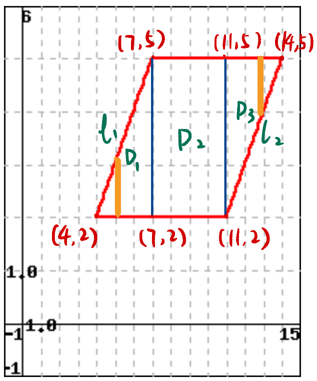
- If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries , then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

See Example 1 for an application

Example 1.

The region R is shown in the figure. Find the limits of integration.



For the format of ① (type 1 in Lecture 14).

We need to divide R into 3 regions. D_1 D_2 D_3

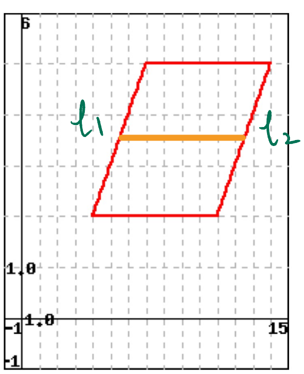
- For D_1, the line ℓ_1: $\frac{y-2}{x-4} = \frac{5-2}{7-4} \Rightarrow y = x-2$
- For D_3, the line ℓ_2: $\frac{y-2}{x-11} = \frac{5-2}{14-11} \Rightarrow y = x-9$

Thus we can fill in the blank as following

$$① \iint_R f(x,y)dA = \underbrace{\int_4^7 \int_2^{x-2} f(x,y)dydx}_{\text{for triangle } D_1} + \underbrace{\int_7^{11} \int_2^5 f(x,y)dydx}_{\text{for rectangle } D_2} + \underbrace{\int_{11}^{14} \int_{x-9}^5 f(x,y)dydx}_{\text{for triangle } D_3}$$

and

$$② \iint_R f(x,y)dA = \int_2^5 \int_{y+2}^{y+9} f(x,y)dxdy$$



type 2 in Lecture 14

For the format of ②, we have

ℓ_1 $\underline{h_1(y)} \le x \le \underline{h_2(y)}$ ℓ_2

$2 \le y \le 5$

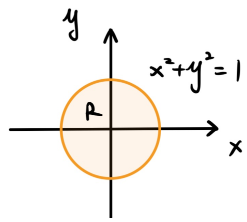
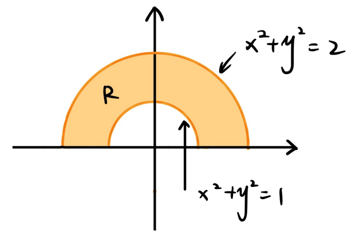
From the above discussion, we know

ℓ_1: $y = x-2 \Rightarrow x = y+2 = h_1(y)$

ℓ_2: $y = x-9 \Rightarrow x = y+9 = h_2(y)$

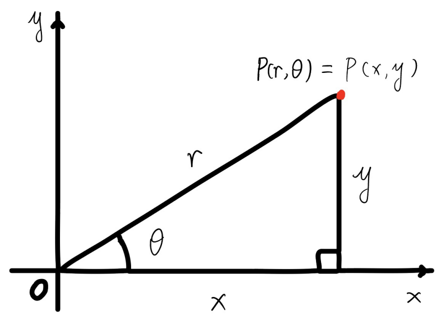
Double Integrals in Polar Coordinates

Suppose that we want to evaluate $\iint_R f(x, y)dA$, where R is one of the regions in the figures below.

$R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$	$R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$
	

- In both cases, while expressing R using rectangular coordinates is somewhat complicated, describing R through polar coordinates simplifies the task.
- Recall from **Lecture 1** that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

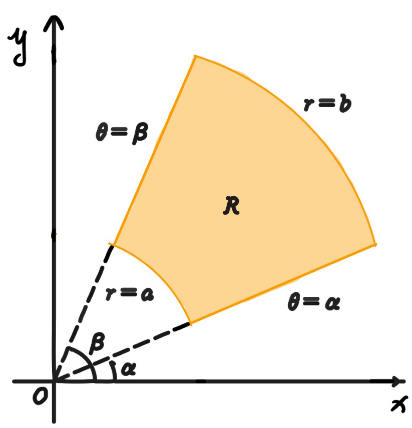
$$r^2 = x^2 + y^2 \quad x = r\cos\theta \quad y = r\sin\theta$$



- The regions in the above table are special cases of a *polar rectangle*

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta\}$$

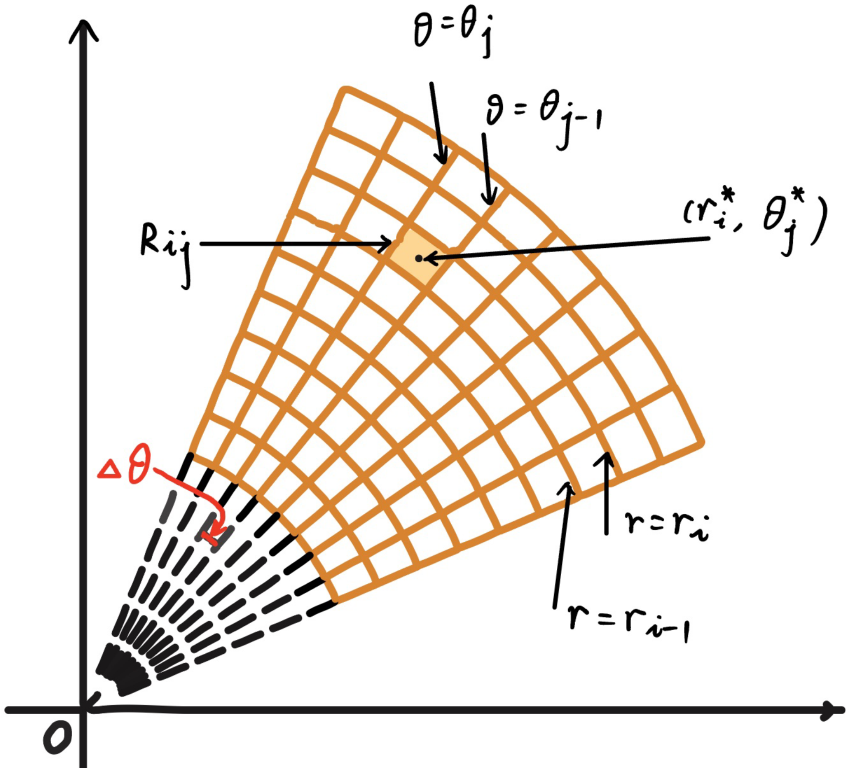
the the following figure.



Polar rectangle

- In order to compute the double integral $\iint_R f(x,y)dA$, where R is a polar rectangle, we divide the interval $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b-a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$.

- Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles shown in the following figure.



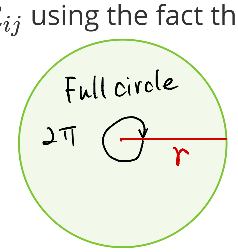
- The midpoint of the polar subrectangle

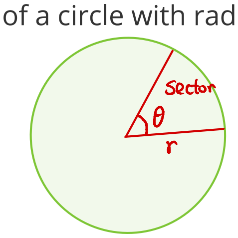
$$R_{ij} = \{(r,\theta) \mid r_{i-1} \leqslant r \leqslant r_i, \theta_{j-1} \leqslant \theta \leqslant \theta_j\}$$

has polar coordinates

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \quad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$$

- We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$.

Recall 

, thus 

Area of Circle $= \pi r^2$ Area of Sector $= \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$

- Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

$$\Delta A_i = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta$$
$$= \frac{1}{2}\left(r_i + r_{i-1}\right)\left(r_i - r_{i-1}\right)\Delta\theta = r_i^*\Delta r\Delta\theta$$

- Although the double integral $\iint_R f(x,y)dA$ in terms of ordinary rectangles, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles.

- The rectangular coordinates of the center of R_{ij} are $\left(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*\right)$, so a typical Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*\right)\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*\right)r_i^*\Delta r\Delta\theta \tag{1}$$

- If we write $g(r,\theta) = rf(r\cos\theta, r\sin\theta)$, then the Riemann sum in Equation (1) can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g\left(r_i^*, \theta_j^*\right)\Delta r\Delta\theta$$

which is a Riemann sum for the double integral

$$\int_\alpha^\beta\int_a^b g(r,\theta)drd\theta$$

- Thus we have

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*\right)\Delta A_i$$
$$= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} g\left(r_i^*, \theta_j^*\right)\Delta r\Delta\theta = \int_\alpha^\beta\int_a^b g(r,\theta)drd\theta$$
$$= \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

The above discussion leads to the following theorem.

Theorem 1. Change to Polar Coordinates in a Double Integral

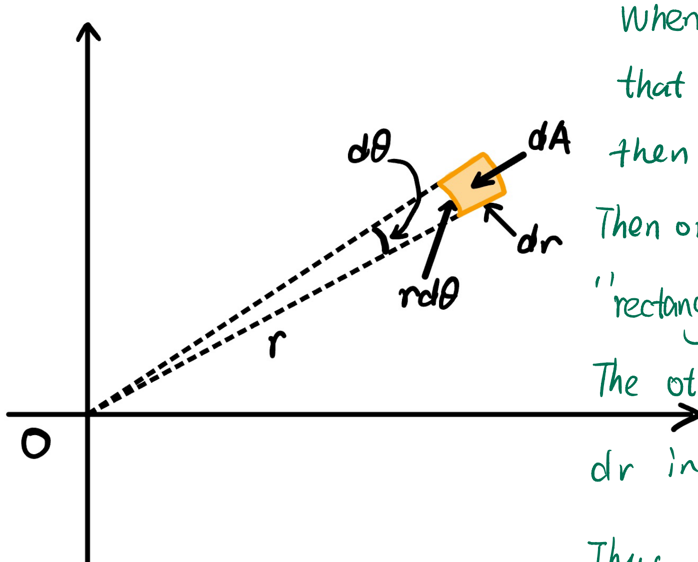
If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\underline{r}\, dr\, d\theta \tag{2}$$

Remark.

- Equation (2) says that we can convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the

  appropriate limits of integration for r and θ, and replacing dA by $rdrd\theta$.
- **Don't forget the additional factor r on the right side of Equation (2).**
- A method for remembering this is shown in the figure below, where the "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $rd\theta$ and dr and therefore has "area" $dA = rdrd\theta$.



When θ is small enough, that is, when $\theta \approx 0$, then $\theta \approx \sin\theta$

Then one side of the "rectangle" has length $\approx rd\theta$

The other side has length dr in an obvious way.

Thus $dA = rdrd\theta$.

Example 2. Suppose R is the shaded region in the figure. As an iterated integral in polar coordinates,

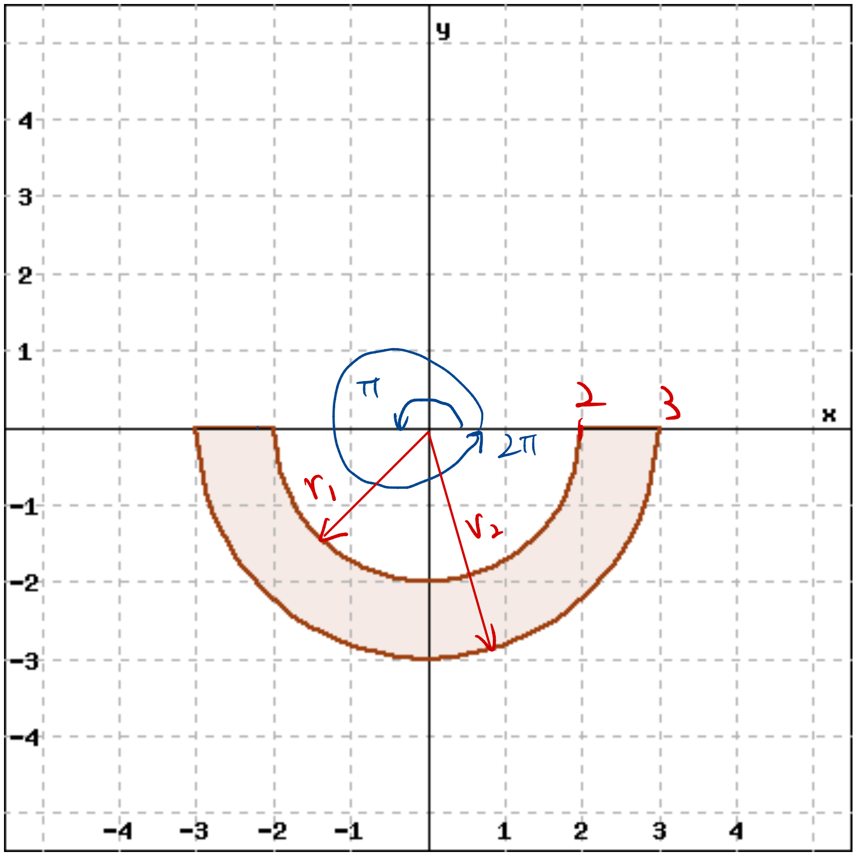
$$\iint_R f(x,y)dA = \int_A^B \int_C^D f(r\cos(\theta), r\sin(\theta))rdrd\theta$$

What are the values for A, B, C and D?

Note θ is in the range from π to 2π

r is in the range.

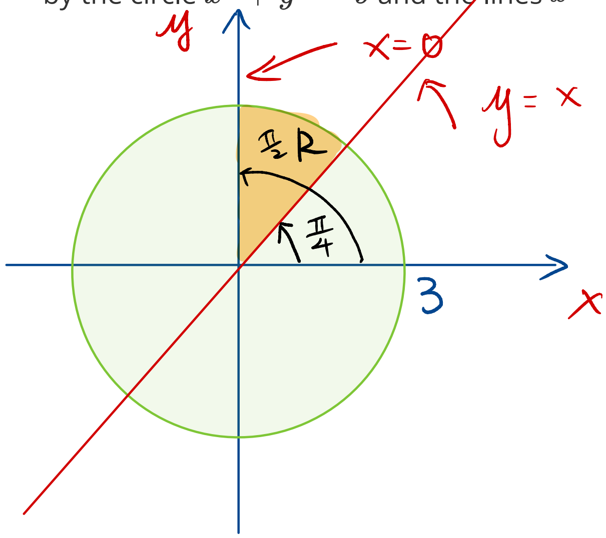
2 to 3.



Thus $A=\pi$, $B=2\pi$

$C=2$, $D=3$.

Example 3. Evaluate the double integral $\iint_R (3x - y)\,dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 9$ and the lines $x = 0$ and $y = x$, by changing to polar coordinates.



We draw the figure on the left.

R. is shaded region.

So the range of r and θ are

$$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le 3$$

Therefore, let $x = r\cos\theta$, $y = r\sin\theta$, by Thm 1, we have

$$\iint_R (3x-y)\,dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^3 (3r\cos\theta - r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^3 3r^2\cos\theta - r^2\sin\theta \,dr\,d\theta$$

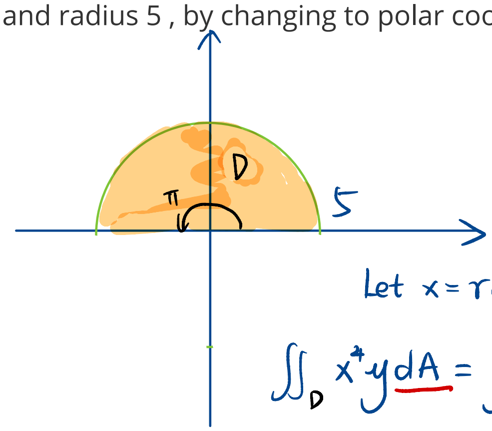
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \Big[r^3\cos\theta - \frac{1}{3} r^3 \sin\theta \Big]_{r=0}^{r=3} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 27\cos\theta - 9\sin\theta \,d\theta$$

$$= \Big[27\sin\theta + 9\cos\theta \Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 27 - \frac{27}{\sqrt{2}} - \frac{9}{\sqrt{2}} = 27 - \frac{36}{\sqrt{2}}$$

$$= 27 - 18\sqrt{2}$$

Example 4. Evaluate the double integral $\iint_D x^4 y\,dA$, where D is the top half of the disc with center the origin and radius 5 , by changing to polar coordinates.



From the figure, we know the range of r and θ are

$$0 \le r \le 5$$
$$0 \le \theta \le \pi$$

Let $x = r\cos\theta$, $y = r\sin\theta$. we know.

$$\iint_D x^4 y\,\underline{dA} = \int_0^{\pi}\int_0^5 r^4\cos^4\theta\; r\sin\theta\;\underline{r\,dr\,d\theta}$$

$$= \int_0^{\pi}\int_0^5 r^6\cos^4\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi}\cos^4\theta\sin\theta\int_0^5 r^6\,dr\,d\theta \quad \text{(factor out the constant)}$$

$$= \int_0^{\pi}\cos^4\theta\sin\theta\;\frac{1}{7}r^7\Big|_0^5\,d\theta$$

$$= \frac{5^7}{7}\int_0^{\pi}\cos^4\theta\sin\theta\,d\theta. \quad ⊛$$

To compute $\int_0^{\pi}\cos^4\theta\sin\theta\,d\theta$, we first find the antiderivative $\int\cos^4\theta\sin\theta\,d\theta$

$$= -\int\cos^4\theta\,d\cos\theta$$

$$= -\frac{1}{5}\cos^5\theta + C$$

(double-check this is right by taking derivative

$$\left(-\frac{1}{5}\cos^5\theta\right)' = +\cos^4\theta\cdot(+\sin\theta)\ \checkmark$$

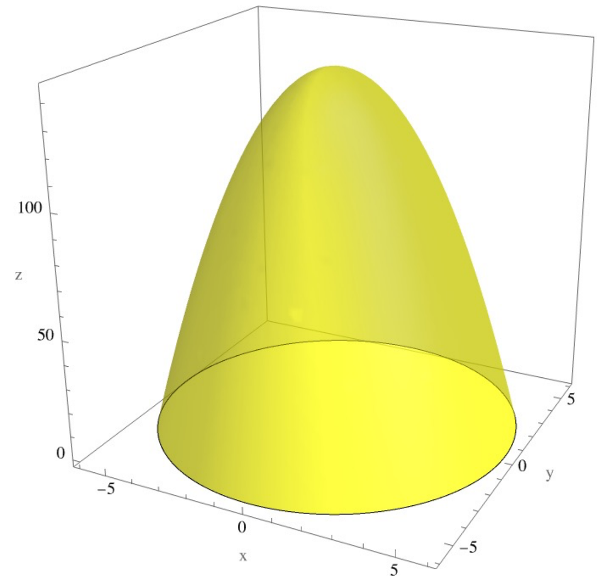
Thus ⊛ becomes

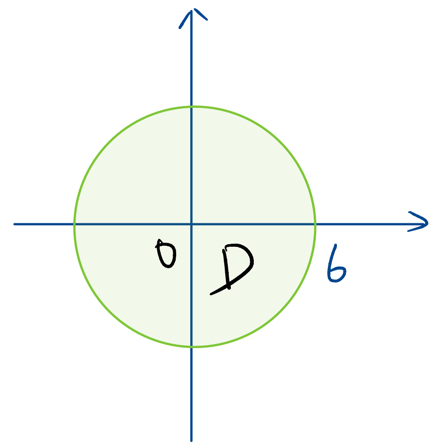
$$⊛ = \frac{5^7}{7}\left(-\frac{1}{5}\cos^5\theta\right)\Big|_0^{\pi}$$

$$= -\frac{5^6}{7}\left[\cancelto{-1}{\cos^5\pi} - \cancelto{1}{\cos^5 0}\right]$$

$$= \frac{2\cdot 5^6}{7} \quad \text{or} \quad \frac{31250}{7}$$

Example 5. Use polar coordinates to find the volume of the solid below the paraboloid $z = 144 - 4x^2 - 4y^2$ and above the xy-plane.





Set $z = 0$. (to check the intersection of the solid with the xy-plane).

we have $144 - 4x^2 - 4y^2 = 0 \Rightarrow x^2 + y^2 = 36 = 6^2$

Thus the intersection of the paraboloid with xy-plane is a circle of radis 6. as the RHS figure above.

Thus it's easier to compute the volumn V using the polar coordinates. We have

$$V = \iint_D 144 - 4x^2 - 4y^2 \, dA \quad (\text{Let } x = r\cos\theta,\ y = r\sin\theta)$$

$$= \int_0^{2\pi} \int_0^6 144 - 4r^2\cos^2\theta - 4r^2\sin^2\theta \; r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^6 144 - 4r^2 (\overset{1}{\cos^2\theta + \sin^2\theta}) \, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^6 144r - 4r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{144}{2} r^2 - \frac{4}{4} r^4 \right]_0^6 d\theta$$

$$= \int_0^{2\pi} \left[72r^2 - r^4 \right]_0^6 d\theta$$

$$= \left(72 \cdot 36 - 6^4\right) \theta \Big|_0^{2\pi}$$

$$= 1296 \cdot 2\pi$$

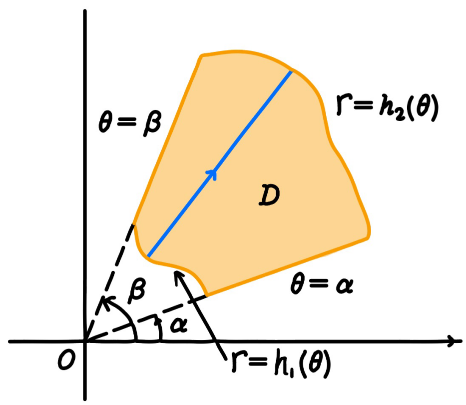
$$\Rightarrow V = 2592\pi$$

If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta$$

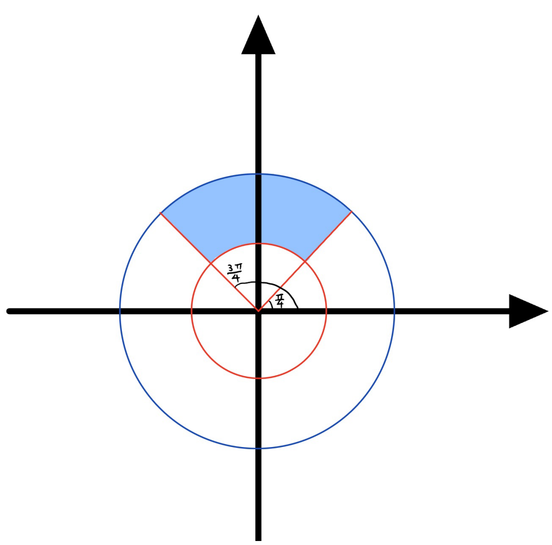


$D = \{(r,\theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$

Exercise 6. Sketch the region whose area is given by the integral and evaluate it.

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_3^6 r dr d\theta$$

Answer.



$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_3^6 r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{r^2}{2}\bigg|_3^6 d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{27}{2} d\theta = \frac{27\theta}{2}\bigg|_{\pi/4}^{3\pi/4} = \frac{27(3\pi)}{2\times 4} - \frac{27\pi}{2\times 4} = \frac{27\pi}{4}.$$

Exercise 7. Evaluate the double integral $\iint_D \cos\sqrt{x^2+y^2}dA$, where D is the disc with center the origin and radius 4 , by changing to polar coordinates.

Answer.

By the description of the question, we know $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$.

$$\iint_D \cos\sqrt{x^2+y^2}dA = \int_0^{2\pi}\int_0^4 r\cos\, r\, dr\, d\theta$$

To compute $\int_0^4 r\cos\, r\, dr$, we first compute the antiderivitive $\int r\cos\, r\, dr$.

Use integration by parts $\int udv = uv - \int vdu$, where

$$\begin{aligned} u = r, &\quad dv = \cos(r)dr \\ du = dr, &\quad v = \sin(r). \end{aligned}$$

Then

$$\int r\cos\, r\, dr = r\sin r - \int \sin r dr = r\sin r + \cos r.$$

Thus $\int_0^4 r\cos\, r\, dr = [r\sin r + \cos r]|_0^4 = -1 + 4\sin(4) + \cos(4)$.

Then

$$\int_0^{2\pi}\int_0^4 r\cos\, r\, dr\, d\theta = \int_0^{2\pi}(-1+4\sin(4)+\cos(4))\, d\theta$$
$$= (\theta(-1+4\sin(4)+\cos(4)))|_0^{2\pi} = 2\pi(-1+4\sin(4)+\cos(4))$$

Exercise 8. Convert the integral

$$I = \int_0^{3/\sqrt{2}} \int_y^{\sqrt{9-y^2}} e^{6x^2+6y^2} dxdy$$

to polar coordinates, getting

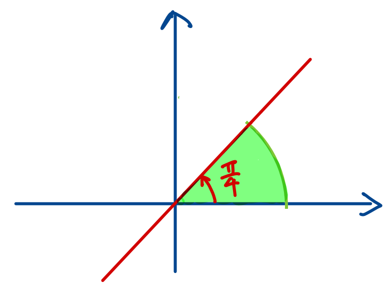
$$\int_C^D \int_A^B h(r,\theta) dr d\theta$$

(a) What are the values of $h(r,\theta)$, A, B, C and D?

(b) Evaluate the value of I.

Answer.

(a) The given integral is equal to the double integral $\iint_D e^{6x^2+6y^2} dA$, where D is the region defined by $0 \le y \le \dfrac{3}{\sqrt{2}}$ and $y \le x \le \sqrt{9-y^2}$; it is the lower half of the quarter-disk of radius 3 in the first quadrant described as the following figure.



Therefore, $h(r,\theta) = re^{6r^2}$, and

$$I = \iint_D e^{6x^2+6y^2} dA = \iint_{D^*} re^{6r^2} dA^*$$

where D^* is defined by $0 \le r \le 3$ and $0 \le \theta \le \pi/4$.

$0 \le \theta \le \frac{\pi}{4}$

$0 \le r \le 3$

(b) From the discussion above, we know

$$\int_0^{3/\sqrt{2}} \int_y^{\sqrt{9-y^2}} e^{6x^2+6y^2} dxdy = \int_0^{\pi/4} \int_0^3 re^{6r^2} dr d\theta$$

$$= \frac{1}{12} \int_0^{\pi/4} \int_0^3 e^{6r^2} d(6r^2) d\theta = \frac{1}{12} \int_0^{\pi/4} e^{6r^2} |_0^3 \, d\theta$$

$$= \frac{1}{12} \int_0^{\pi/4} (e^{54} - 1) d\theta = \frac{1}{12} (e^{54} - 1)\theta \Big|_0^{\pi/4} = \frac{1}{48} (e^{54} - 1)\pi.$$